

Stable degenerations of surfaces isogenous to a product of curves

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1 Introduction

In [Cat00] and its sequel, [Cat03], Catanese studies surfaces which admit an unramified cover by a product of curves of genus greater than one. Then main interesting result from the point of view of moduli theory is that the moduli space of such surfaces (that is, of canonically polarized surfaces with the same numerical invariants K^2 and $\chi(\mathcal{O})$) is either irreducible or has two connected components, swapped by complex conjugation.

In [KSB88], Kollár and Shepherd-Barron introduced a compactification of the moduli space of canonically polarized surfaces. The details of the construction were later filled in by several authors. In particular, they give a *stable reduction* procedure, by which any one-parameter family of surfaces over a punctured disk can be completed to a family of so-called *stable surfaces* over a finite cover of the disk. The stable reduction is obtained by taking the relative canonical model of a semistable resolution of the original family.

This article is concerned with determining the stable surfaces which occur at the boundary of this moduli space. The moduli space of stable curves provides us with candidates for the boundary surfaces, and we simply verify that these candidates are already stable, so the full force of the minimal model program is not required. In contrast, a forthcoming article demonstrates the use of flips in determining the stable degenerations of symmetric squares of curves.

Pairing our results with Catanese's results, we obtain that the two components of the moduli space of canonically polarized surfaces do not meet in the stable compactification. To my knowledge, this is the first example of disconnectedness of a moduli space of *stable* surfaces after fixing the diffeomorphism class of a smooth member. The disconnectedness of the moduli space of canonically polarized surfaces with fixed differentiable structure was first proved by Manetti. However, to establish that his surfaces which lie on different components of the moduli space are diffeomorphic, he degenerates them all to a single stable surface. Catanese's examples, on the other hand, are obviously diffeomorphic surfaces.

2 Stable surfaces and surfaces isogenous to a product

First we define the higher-dimensional analogue of the nodes which are allowed on stable curves:

Definition 2.1. A surface S has *semi-log canonical* (slc) singularities if

1. S is Cohen-Macaulay;
2. S has normal crossings singularities in codimension one;
3. S is \mathbf{Q} -Gorenstein, i.e. some reflexive power of the dualizing sheaf of S is a line bundle;
4. for any birational morphism $\pi : X \rightarrow S$ from a smooth variety, if we write (numerically)

$$K_X = \pi^* K_S + \sum a_i E_i,$$

then all the $a_i \geq 1$.

The first condition ensures that the dualizing sheaf exists; the second gives that it is an invertible sheaf off a subset of codimension 2, so it can be extended to give a Weil divisor class K_S . The third condition states that some multiple of this class is Cartier, so we can make sense of the formula occurring in the fourth condition. A complete classification of slc surface singularities can be found in [KSB88]. Suppose S is a \mathbf{Q} -Gorenstein surface. Let S^\vee be the normalization, and D the inverse image of the codimension 1 singular set under the normalization morphism. Then the condition that S be slc is equivalent to the condition that (S^\vee, D) is a log canonical (lc) pair.

Definition 2.2. A *stable surface* is a projective, reduced surface S with slc singularities such that some reflexive power $\omega_S^{[N]}$ is an ample line bundle. The smallest such N such that $\omega_S^{[N]}$ is a line bundle is called the *index* of S . A family of stable surfaces is a flat morphism $X \rightarrow B$ whose fibers are stable surfaces and whose relative dualizing sheaf $\omega_{X/B}$ is \mathbf{Q} -Cartier.

After fixing a Hilbert polynomial P , there is a bound for the index of a stable surface with Hilbert polynomial P . This is one of the ingredients in the construction of the moduli space of stable surfaces with fixed Hilbert polynomial P , which compactifies the moduli space of canonically polarized surfaces (after possibly throwing away some components parameterizing only singular surfaces). This moduli space is proper and separated. The moduli space would not be separated without the requirement that families of stable surfaces be relatively \mathbf{Q} -Gorenstein. It is worth noting that in a later article ([Kol90]) Kollár strengthened the conditions that a family of stable surfaces should satisfy. It is unknown whether the \mathbf{Q} -Gorenstein assumption alone implies these stronger condition. For our purposes, the weaker condition is sufficient, since it is known to imply the stronger conditions in the case of one-parameter families whose general member is smooth.

We now recall some of the definitions from [Cat00].

Definition 2.3. A surface S is called *isogenous to a product* if it admits an unramified cover by a product of curves of genus two or higher.

Remark 2.4. Catanese calls such a surface *isogenous to a higher product*, but we will not be interested in surfaces covered by, say, a product of elliptic curves. It is clear that any such surface is canonically polarized, since the cover by a product of curves contains no rational curves, so S contains none.

The following is a summary of 3.10-3.13 of *loc. cit.*:

Proposition 2.5. A surface S isogenous to a product can be written uniquely as $(C_1 \times C_2)/G$ for some group G which embeds into $\text{Aut } C_1$ and $\text{Aut } C_2$, as long as C_1 and C_2 are not isomorphic. If C_1 and C_2 are isomorphic, the subgroup of G consisting of automorphisms not switching the factors is required to embed into the automorphism group of each factor to obtain this minimal realization.

This proposition allows us to describe small deformations of surfaces isogenous to a product. This was done in *loc. cit.*, but we present a simple and purely algebraic proof here.

Definition 2.6. The functor of deformations of a stable curve C with the action of a finite group G assigns to an artin ring A :

1. a flat morphism $X \rightarrow \text{Spec } A$,
2. an embedding of G in the group $\text{Aut}_{\text{Spec } A} X$ of automorphisms of the family over the base,
3. and an equivariant isomorphism of the special fiber of the family X with the stable curve C .

Proposition 2.7. The functor of deformations of a stable curve C together with a subgroup G of the automorphism group is “well-behaved” (i.e. satisfies the Schlessinger conditions) and unobstructed, with tangent space $\text{Ext}^1(\Omega_C, \mathcal{O}_C)^G$. Such pairs have a proper moduli space, finite over a closed subvariety of the moduli space of stable curves.

The proof of this may be found in [Tuf93]. Note that the statements here depend on the exactness of taking invariants by a finite group in characteristic zero, and that the moduli space mentioned above is not proper in positive characteristic because of wild ramification.

Proposition 2.8. The Kuranishi space of a surface S minimally realized as a free quotient $(C_1 \times C_2)/G$ is (locally analytically) isomorphic to the product of Kuranishi spaces of the pairs (C_1, G) and (C_2, G) . Consequently, the Kuranishi space is smooth, and the moduli space of canonically polarized surfaces is irreducible at S .

Proof. Since all deformation functors in question are “nice” enough, we may resolve the question by checking to first order:

$$\begin{aligned} H^1(S, \mathcal{T}_S) &= H^1(C_1 \times C_2, \mathcal{T}_{C_1 \times C_2})^G \\ &= H^1(C_1, \mathcal{T}_{C_1})^G \oplus H^1(C_2, \mathcal{T}_{C_2})^G. \end{aligned}$$

The first line is valid only when G acts freely, but the second line works generally for products of canonically polarized varieties, assuming G acts on both factors (see e.g., [vO]).

Since deformations of curves with group action are unobstructed, the Kuranishi space of S is smooth. Since S has a finite automorphism group, the moduli space of S locally near S has only finite quotient singularities, so cannot be reducible. \square

3 Degenerations

Our main goal is the following theorem:

Theorem 3.1. *Suppose $X \rightarrow \Delta'$ is a family of surfaces isogenous to products over a punctured disk. Then possibly after a finite change of base, totally ramified over the origin in the disc, X (or a pullback thereof) can be completed to a family of stable surfaces over the disk whose central fiber is a quotient of a product of stable curves (possibly by a non-free group action).*

Proof. By 2.8, we may assume that X is of the form $(Y_1 \times_{\Delta'} Y_2)/G$ where Y_1 and Y_2 are families of smooth curves with G -action, such that the G -action is fiberwise and free on $Y_1 \times_{\Delta'} Y_2$. Since the moduli functor of stable curves with automorphism group G is proper, after a base change (which we will suppress in our notation), we obtain a family \tilde{X} of the desired form.

It remains to see that the central fiber is a stable surface, and that the family \tilde{X} is a family of stable surfaces. \tilde{X} is obtained by taking the quotient of a family \tilde{Y} of stable surfaces by a group action. Since the group acts freely on the general fiber, the quotient morphism is étale in codimension 1. In this case, [KM98], Proposition 5.20 ensures that \tilde{X} is \mathbf{Q} -Gorenstein, so the special fiber is as well. Well known results ensure that the special fiber is Cohen-Macaulay. A finite quotient of a variety which is normal crossings in codimension one is normal crossings in codimension one by Corollary 1.7 of [AAL81]. Now [KM98] 5.20 (appropriately modified to take into account non-normal varieties) states that a finite quotient of an slc variety is slc as soon as it is \mathbf{Q} -Gorenstein an normal crossings in codimension 1.

Finally, since the relative canonical sheaf $\omega_{\tilde{Y}/\Delta}$ is ample since Y is a family of products of curves of general type, and the quotient morphism is unramified in codimension one, the relative canonical sheaf of \tilde{X} is also ample, so \tilde{X} is a family of stable surfaces, hence the special member is uniquely determined by separatedness of the moduli space of stable surfaces. \square

4 Digression: Deformation of curves with group action

For a family $X \rightarrow S$ of proper varieties, the functions $\dim T^n(X_s)$ are Zariski upper semicontinuous; that is, they may jump up at certain points. This is proved in [Pal86], where actually a stronger result is obtained:

Theorem 4.1 (Palamodov). *Let $X \rightarrow S$ be a flat proper morphism. The functions $\sum_{i=0}^{2m} (-1)^i \dim T^{n+i}(X_s)$ are upper semicontinuous, for $m > 0$ and $n \geq 0$.*

A consequence for stable curves is that $\dim T^1(C_s)$ is constant in a family of stable curves, since T^0 and T^2 vanish for stable curves. This is one way of showing that even for a stable curve, $T^1(C)$ is $3g - 3$ -dimensional, although direct computation is straightforward, even for an arbitrary nodal curve.

Does such a strong semicontinuity property hold for the functors $T^i(-, G)$? A weak version follows from Palamodov's theorem, at least in our situation of finite groups and varieties defined over fields of characteristic zero:

Lemma 4.2. *Let $X \rightarrow S$ be a family of varieties with G -action. Then $\dim T^i(X_s, G)$ is an upper semicontinuous function of $s \in S$.*

Proof. From [Pal86], we have upper semicontinuity for $\dim T^i(X_s)$. Under our assumptions, $T^i(X_s, G) = T^i(X_s)^G$. The set of G -invariants is the intersection of the kernels of the linear maps $g - I$ for all $g \in G$, and it is well known that the dimension of the kernel of a vector bundle map is upper semicontinuous. \square

A review of the computation of $T^1(C)$ for a stable curve will be useful in computing $T^1(C, G)$. First of all, for nodal curves (or more generally, complete intersection curves), the local-to-global Ext spectral sequence computes

T^i : $T^i(C) = 0$ for $i > 1$, $T^0(C) = H^0(C, \mathcal{T}_C)$, and we can write, using the exact sequence of low degree terms of this spectral sequence,

$$0 \longrightarrow H^1(\mathcal{T}_C) \longrightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C) \longrightarrow H^0(C, \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)) \longrightarrow 0$$

an extension describing T^1 . Let us describe the topological data of a stable curve as in [HM98]: let v be the number of irreducible components. Then let g_i be the genus of the normalization of the i th irreducible component for $i = 1, \dots, v$. Finally, let δ equal the number of nodes. Then the (arithmetic) genus of C is given by

$$g = \sum_{i=1}^v g_i + \delta - v + 1.$$

The space $H^0(C, \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C))$ is easy to compute: the sheaf $\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C)$ is supported at the nodes with one-dimensional stalks. Therefore, the dimension of $H^0(C, \mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C))$ is δ .

For a nodal curve C , let C^\vee be the normalization. Then there are pairs (p_k, q_k) of points for $k = 1, \dots, \delta$ lying over each node and giving an exact sequence of sheaves on C^\vee :

$$0 \longrightarrow \mathcal{T}_{C^\vee}(-\sum_{k=1}^\delta (p_k + q_k)) \longrightarrow \mathcal{T}_{C^\vee} \longrightarrow \mathcal{T}_{C^\vee} \otimes \mathcal{O}_{\sum_{k=1}^\delta (p_k + q_k)}^\delta \longrightarrow 0$$

By stability, $H^0(\mathcal{T}_{C^\vee}(-\sum_{k=1}^\delta (p_k + q_k))) = 0$, and for dimension reasons, $H^1(\mathcal{T}_{C^\vee} \otimes \mathcal{O}_{\sum_{k=1}^\delta (p_k + q_k)}^\delta) = 0$. Therefore

$$\dim T^1(C) = \delta + h^0(\mathcal{T}_{C^\vee} \otimes \mathcal{O}_{\sum_{k=1}^\delta (p_k + q_k)}^\delta) - \chi(\mathcal{T}_{C^\vee}),$$

where we use the fact that $H^1(C, \mathcal{T}_C) = H^1(C^\vee, \mathcal{T}_{C^\vee}(-\sum_{k=1}^\delta (p_k + q_k)))$.

The pieces are easy to compute:

$$-\chi(\mathcal{T}_{C^\vee}) = 3 \sum_{i=1}^v g_i - 3v$$

and

$$h^0(\mathcal{T}_{C^\vee} \otimes \mathcal{O}_{\sum_{k=1}^\delta (p_k + q_k)}^\delta) = 2\delta.$$

We conclude that

$$\dim T^1(C) = 3 \sum_{i=1}^v g_i + 3\delta - 3v$$

which is $3g - 3$ in light of the formula given above for g .

Now to the equivariant case: since the operation of taking G -invariants is exact, we obtain a similar formula:

$$\dim T^1(C, G) = h^0(\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C))^G + h^0(\mathcal{T}_{C^\vee} \otimes \mathcal{O}_{\sum_{k=1}^\delta (p_k + q_k)}^\delta)^G - \chi(\mathcal{T}_{C^\vee})^G$$

where by $\chi(\mathcal{T}_{C^\vee})^G = \dim H^1(\mathcal{T}_{C^\vee})^G - \dim H^0(\mathcal{T}_{C^\vee})^G$. Note that the G -action extends to C^\vee , since G acts birationally on C^\vee (normalization is birational) and C^\vee is a smooth curve, so any birational morphism extends to an automorphism.

The following example illustrates the method of computation in the case where G does not permute the components of C^\vee . In this case, C^\vee/G is a disjoint union of smooth curves. We also make use of the fact that the deformation functor $\text{Def}_{(C, G)}$ of a smooth curve with group action is isomorphic to the functor $\text{Def}_{(C/G, B)}$ where B is the branch locus of the cover $C \rightarrow C/G$.

Example 4.3. Let C be the curve defined by the homogenous equation $xyz^2 + x^4 + y^4 = 0$ in \mathbf{P}^2 . The arithmetic genus of C is 3, and C has a single node at the point with homogeneous coordinates $(0 : 0 : 1)$. The surface $xyz^2 + x^4 + y^4 + tz^4$ in $\mathbf{P}_{A_1}^2$ is a one-parameter smoothing.

Let $G = \mathbf{Z}_2$ act on C by swapping x and y and sending z to $-z$. The fixed locus of this automorphism of \mathbf{P}^2 is the set of points with homogeneous coordinates $(1 : -1 : z)$ and the points $(0 : 0 : 1)$ and $(1 : 1 : 0)$, that is, a line together with an isolated point. Therefore, G acts on a general member of the smoothing (a smooth quadric) fixing four points. The action of G on C fixes three points, one of them the node.

Let C^\vee denote the normalization of C and p and q be the two points of C^\vee lying over the node of C . The group action fixes the node of C , but swaps the two branches. This means that the action lifted to C^\vee swaps p and q . Analytically locally near the node, the curve looks like $\mathbb{C}[x, y]/(xy)$ together with the action of G swapping x and y . Since the G action lifts to a versal deformation, we conclude that every element of $H^0(\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C))$ is G -invariant. This vector space is one-dimensional.

Next we compute $H^0(\mathcal{O}_{p,q} \otimes \mathcal{T}_{C^\vee})^G$. The space $H^0(\mathcal{O}_{p,q} \otimes \mathcal{T}_{C^\vee})$ is a two-dimensional vector space. Since the G -action on C^\vee swaps p and q , only the “diagonal” (in an appropriate basis) is fixed by G . So this space contributes one dimension to $T^1(C, G)$.

Finally, we need to compute $\chi(\mathcal{T}_{C^\vee})^G$. C^\vee is a genus 2 curve (genus drops by the number of nodes upon normalization) and it double covers C^\vee/G with two points fixed (the two points of C other than the node which are fixed by G). Therefore, C^\vee/G is an elliptic curve with two marked points. It follows that $\chi(\mathcal{T}_{C^\vee})^G = -2$, and therefore $T^1(C, G)$ is four-dimensional.

In this example, there is no jumping of dimension: the general fiber is a genus 3 curve with G -action whose quotient is a 4-pointed elliptic curve. Therefore T^1 of the generic fiber with G -action is also four-dimensional.

Note that it is critical how G acts near a node in order to compute $H^0(\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C))^G$; it is not the case that if G fixes a node, then every element of $H^0(\mathcal{E}xt^1(\Omega_C^1, \mathcal{O}_C))$ is fixed.

Now we show that the example is general: $\dim T^1(C_s, G)$ cannot jump in a family of stable curves.

Theorem 4.4. *Let $C \rightarrow S$ be a family of stable curves with an action of a finite group G . Then $\dim T^1(C_s, G)$ is locally constant.*

Proof. First of all, the deformation functor of curves of genus greater than one with group action is formally smooth (this is true even in positive characteristic, assuming that the action is tame, by [Tuf93], 5.1). This functor has a versal deformation, which is universal, since $T^0(C, G) = 0$ for stable curves. The universal family induces a morphism to the moduli space, and the induced morphism is finite and surjective onto a neighborhood of the point of the moduli space corresponding to the isomorphism class of (C_0, G) . Therefore the moduli space has at worst finite quotient singularities, and its dimension is equal to $\dim T^1(C_0, G)$. Since the moduli space is locally irreducible, its dimension cannot jump at special points, from which the theorem follows. \square

5 Application to Catanese’s examples

In [Cat03], Catanese gives a family of examples of moduli spaces of smooth surfaces with fixed K^2 and χ fixed which have two components interchanged by complex conjugation. We review his construction here and study the degenerations of his surfaces.

The construction of the example begins with the construction of a triangle curve (i.e. a Galois cover of \mathbb{P}^1 branched at 3 points) which is not antiholomorphic to itself. Let C denote this curve and G denote the Galois group of the cover $C \rightarrow \mathbb{P}^1$. G is therefore a quotient of the fundamental group of \mathbb{P}^1 minus three points, and is consequently generated by 2 elements. Choose $h \geq 2$ and a curve C'_1 of genus h . Then the fundamental group of C'_1 surjects onto G , so there exists an étale cover $C_1 \rightarrow C'_1$ with Galois group G . Then the surface $S = (C_1 \times C)/G$ is isogenous to a product of curves of general type (the triangle curve constructed is not the elliptic curve with j -invariant 1728, which is the only triangle curve not of general type).

The critical result for finding multiple components of the moduli space is Catanese’s Proposition 3.2: the existence of an antiholomorphic isomorphism of two surfaces minimally realized as surfaces isogenous to products of curves of general type implies antiholomorphic isomorphisms of the factors (up to reordering the factors). In what follows, denote by \overline{X} the complex conjugate of the manifold X .

Choosing any C'_2 of genus h and a cover C_2 of C'_2 with Galois group G as above, suppose $(C_1 \times C)/G \cong \overline{(C_2 \times C)/G}$. Then there is an antiholomorphic isomorphism of $(C_1 \times C)/G$ with $(C_2 \times C)/G$, and hence, an antiholomorphic automorphism of C , which is impossible by the construction of C . Catanese claims that the various choices of C_2 fill out a component of the moduli space. But $\overline{(C_2 \times C)/G}$ is diffeomorphic to $(C_2 \times C)/G$, and hence also has a point in the moduli space, which cannot be on this component. Therefore the moduli space has at least two components.

Now let us consider the stable degenerations of these surfaces, and address the question of whether the two components are joined together by deformations through stable surfaces. The results in this chapter show that the (small) deformations of S are just the G -equivariant deformations of C_1 , or equivalently, the deformations of C_1/G . Let \overline{M}

denote the moduli space of smoothable stable surfaces occurring as degenerations of $(C_1 \times C)/G$ or its conjugate. \overline{M} has two irreducible components; is it connected?

Suppose \overline{M} were connected: then there would exist a surface $(C' \times C)/G$ on the boundary of the moduli space which lies on the closure of both components. Since both components come from curves with G -action, the map induced from the Kuranishi space of $(C' \times C, G)$ must surject onto a neighborhood of the corresponding boundary point. But the Kuranishi space of $(C' \times C, G)$ is irreducible, so it cannot map onto two components. So the disconnection of various moduli spaces considered in [Cat03] continues in the stable compactification.

Note that this argument is not strong enough in general to claim that a moduli space of surfaces isogenous to a product of curves is always irreducible at the boundary: it just rules out deformations to other surfaces isogenous to a product with the same Galois group. By the results of [Cat03], there are at most two components of the moduli space (after fixing Hilbert polynomial) parameterizing smooth varieties, but there may be a component parameterizing only singular surfaces meeting both other components along the boundary.

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